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Theoretical pressure–strain term in a stratified fluid

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The influence of buoyancy on the pressure–strain term is calculated approximately by an analytical theory. It is shown that the buoyancy contribution to $(\phi_{ij}+\phi_{ji})_1$, the fluctuation part of the pressure–strain term, is approximately equal to the buoyancy contribution which comes from the mean-field part of the pressure–strain term, provided that the mean buoyancy does not vary rapidly in space or time. The latter, but not the former, buoyancy contribution was previously obtained by Launder (1975) and by Zeman & Lumley (1976). Both contributions are shown to be accounted for by use of a single numerical coefficient C^*_θ. The value of C^*_θ predicted from purely theoretical considerations is 0.7, and a value determined from an experiment is 0.9. The theoretical method has some generality and can be applied to higher than second-order correlations of velocity and temperature fluctuations.

1. Introduction

In previous papers, the fluctuation field part of the pressure–strain rate was theoretically calculated for unstratified fluids (Weinstock 1981*a*, 1982; hereinafter referred to as I and II). The theory was tested by comparison with nearly homogeneous shear flows for both weak and strong shear strength (Weinstock & Burk 1985). The purpose of our present article is to extend that theory to stratified fluids. The extended theory will then be compared with models of the pressure–strain rate as suggested by Launder (1975), Zeman & Lumley (1976), and Lumley, Zeman & Seiss (1978).

Our method of calculation is based on the cumulant discard used in I for the unstratified case (Weinstock 1981*b*). The discarded cumulant is a two-time correlation, and as such, constitutes a much milder approximation than the discard of single-time correlations in quasi-normal theory (e.g. Proudman & Reid, 1954). The goal is to calculate the pressure–strain term in terms of energy spectra, not to calculate the spectra themselves – a much less ambitious goal than that of other statistical turbulence theories (e.g. the direct interaction approximation, Kraichman 1959). This limited goal allows our calculation to be much simpler than such theories, and is entirely analytic. Furthermore, previous knowledge or experience with contemporary methods of statistical turbulence theory is neither required nor expected. Our plan is to repeat briefly the derivation in I in such a way as to include stratification in a straightforward manner. This derivation determines the contribution of stratification of both parts of the pressure–strain correlation – the nonlinear-fluctuation part (the slow term) and the mean-field part. To our knowledge, the contribution of stratification to the nonlinear-fluctuation part has not been considered previously. A related calculation was recently reported by Dakos & Gibson (1985) for a pressure–scalar correlation; the pressure–strain term was not under consideration.

For further simplification, to help clarify the theory and its underlying approximations we assume ∇U, the mean shear, and all mean values (including correlation functions) vary but little in a space and time on scales L_0 and τ_L respectively, where L_0 is the characteristic lengthscale of the scalar energy spectrum and τ_L is the Lagrangian timescale.

2. Pressure–strain rate

The pressure–strain rate correlation that appears in the stress transport equation (the Reynolds stress equation) is

$$\boldsymbol{\phi}+\boldsymbol{\phi}^{\mathrm{T}} \equiv \rho_0^{-1}\langle p[\nabla \boldsymbol{u}+(\nabla \boldsymbol{u})^{\mathrm{T}}]\rangle, \tag{1}$$

where $\boldsymbol{u} \equiv \boldsymbol{u}(\boldsymbol{x}, t)$ is the fluctuation part of the fluid velocity at spatial position $\boldsymbol{x}$ at time t, $p \equiv p(\boldsymbol{x}, t)$ is the pressure fluctuation, ρ_0 is the mean particle density assumed constant, the angle brackets denote the ensemble average (mean value), and the superscript T denotes the transpose of a dyadic; e.g. the i, j components of $\nabla \boldsymbol{u}$ and $(\nabla \boldsymbol{u})^{\mathrm{T}}$ are $\partial u_j/\partial x_i$ and $\partial u_i/\partial x_j$ respectively, where i and j denote Cartesian coordinates 1, 2 or 3 (i.e. $i = 1, 2$ *or* 3). The fluctuation velocity includes all random fluctuations, random-phased gravity-wave fluctuations as well as turbulence.

To calculate $\langle p\nabla \boldsymbol{u}\rangle$, or $\langle p(\nabla \boldsymbol{u})^{\mathrm{T}}\rangle$, we need expressions for p and $\boldsymbol{u}$, and both quantities can be obtained from the Navier–Stokes equation. The fluctuation part of that equation is given by

$$\frac{\partial \boldsymbol{u}}{\partial t}+(\boldsymbol{u}+U)\cdot\nabla \boldsymbol{u} = \langle \boldsymbol{u}\cdot\nabla \boldsymbol{u}\rangle-\frac{\nabla p}{\rho_0}-\boldsymbol{u}\cdot\nabla U-\frac{\boldsymbol{g}\theta}{\Theta_0}+\nu\nabla^2\boldsymbol{u}, \tag{2}$$

where Θ_0 is the mean (potential) temperature, θ is the fluctuation potential temperature, $\boldsymbol{g}$ is the acceleration due to gravity $[\boldsymbol{g} \equiv (0, -g, 0)]$, and ν is the molecular viscosity.

A formal expression for p is obtained by taking the divergence of (2) and using incompressibility $\nabla\cdot\boldsymbol{u} = 0$.

$$\frac{\nabla^2 p}{\rho_0} = -\nabla\cdot(\boldsymbol{u}\cdot\nabla\boldsymbol{u})'-2\nabla\boldsymbol{u}:\nabla U-\boldsymbol{g}\cdot\nabla\left(\frac{\theta}{\Theta_0}\right), \tag{3}$$

were we have defined $(\boldsymbol{u}\cdot\nabla\boldsymbol{u})' \equiv (\boldsymbol{u}\cdot\nabla\boldsymbol{u})-\langle\boldsymbol{u}\cdot\nabla\boldsymbol{u}\rangle$. Equation (3) can be solved for p by using Fourier transforms. The Fourier transforms of p and $\boldsymbol{u}$ are defined by

$$p(\boldsymbol{k}) \equiv \int \mathrm{d}\boldsymbol{x}\, p \exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}), \quad \boldsymbol{u}(\boldsymbol{k}) \equiv \int \mathrm{d}\boldsymbol{x}\, \boldsymbol{u} \exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}). \tag{4}$$

We then obtain $p(\boldsymbol{k}) \equiv p(\boldsymbol{k}, t)$ from the Fourier transform of (3):

$$\frac{p(\boldsymbol{k})}{\rho_0} = N(\boldsymbol{k})+2\mathrm{i}\boldsymbol{u}(\boldsymbol{k})\cdot\nabla U\cdot\frac{\boldsymbol{k}}{k^2}+\frac{\mathrm{i}\boldsymbol{g}\cdot\boldsymbol{k}}{k^2}\frac{\theta(\boldsymbol{k})}{\theta_0}. \tag{5}$$

Here, $k^2N(\boldsymbol{k})$ denotes the Fourier transform of the nonlinear fluctuation term $\nabla\cdot(\boldsymbol{u}\cdot\nabla\boldsymbol{u})'$ given by

$$N(\boldsymbol{k}) \equiv -\int \frac{\mathrm{d}\boldsymbol{k}_{\mathrm{a}}}{(2\pi)^3}\frac{\boldsymbol{k}\boldsymbol{k}}{k^2}:[\boldsymbol{u}(\boldsymbol{k}_{\mathrm{a}})\,\boldsymbol{u}(\boldsymbol{k}-\boldsymbol{k}_{\mathrm{a}})-\langle\boldsymbol{u}(\boldsymbol{k}_{\mathrm{a}})\,\boldsymbol{u}(\boldsymbol{k}-\boldsymbol{k}_{\mathrm{a}})\rangle], \tag{6}$$

$\theta(\boldsymbol{k})$ is the Fourier transform of θ, we have used our simplifying assumption that $\nabla\boldsymbol{u}$ and θ_0 vary slower in space than do $\nabla\boldsymbol{u}$ and θ respectively, and periodic boundary conditions are assumed in order to avoid the affects of boundaries.

Next we substitute (5) into the Fourier expansion of the pressure–strain correlation $\boldsymbol{\phi} \equiv \rho_0^{-1}\langle p\nabla \boldsymbol{u}\rangle \equiv -(2\pi)^{-3} V^{-1}\int \mathrm{d}\boldsymbol{k}\, \mathrm{i}\boldsymbol{k}\langle \boldsymbol{u}(\boldsymbol{k})^* p(\boldsymbol{k})\rangle$ and obtain three parts as follows:

$$\frac{\langle p\nabla \boldsymbol{u}\rangle}{\rho_0} = \frac{-\mathrm{i}}{V}\int \frac{\mathrm{d}\boldsymbol{k}\boldsymbol{k}}{(2\pi)^3} \times \Big\{\underbrace{\langle \boldsymbol{u}(\boldsymbol{k})^* N(\boldsymbol{k})\rangle}_{\boldsymbol{\phi}_{,1}} + \underbrace{2\mathrm{i}\langle \boldsymbol{u}(\boldsymbol{k})^* \boldsymbol{u}(\boldsymbol{k})\rangle \cdot \nabla \boldsymbol{U}\cdot \frac{\boldsymbol{k}}{k^2}}_{\boldsymbol{\phi}_{,2}} + \underbrace{\frac{\mathrm{i}\boldsymbol{g}\cdot\boldsymbol{k}}{\theta_0}\langle \boldsymbol{u}(\boldsymbol{k})^* \theta(\boldsymbol{k})\rangle}_{\boldsymbol{\phi}_{,3}}\Big\}, \tag{7}$$

where V is the volume of the system, the superscript * denotes the complex-conjugate, and our assumption of slow spatial variations of correlations (quasi-homogeneity) has been used to approximate $\langle \boldsymbol{u}(\boldsymbol{k})^* p(\boldsymbol{k}')\rangle = (2\pi)^3 V^{-1}\delta(\boldsymbol{k}-\boldsymbol{k}')\langle \boldsymbol{u}(\boldsymbol{k})^* p(\boldsymbol{k})\rangle$ in $\boldsymbol{\phi}$. If (7) appears different from the $\boldsymbol{\phi}$-expressions of Zeman & Lumley (1976), it is only because, here, we use Fourier transforms and quasi-homogeneity. The $\boldsymbol{\phi}_1$ part is due to nonlinear fluctuations (the part associated with tendency-towards-isotropy and referred to as the slow term), the $\boldsymbol{\phi}_{,2}$ part (referred to as the fast term) is due to mean shear, and the $\boldsymbol{\phi}_{,3}$ part is due to mean stratification (buoyancy).

The explicit buoyancy term $\boldsymbol{\phi}_{,3}$ was calculated by Launder (1975) and Zeman & Lumley (1976). However, to our knowledge, the implicit buoyancy correction to $\boldsymbol{\phi}_{,1}$ has not been considered. Our goal is to calculate the buoyancy correction to $\boldsymbol{\phi}_{,1}$, compare it with $\boldsymbol{\phi}_{,3}$, and then combine them so as to obtain a more complete theory for the influence of buoyancy on the pressure–strain term. A buoyancy correction to $\boldsymbol{\phi}_{,2}$ need not be considered since it is given directly in terms of velocity spectra (e.g. Launder, Reece & Rodi 1975; Lumley 1978; Reynolds 1976).

3. Calculation of $\boldsymbol{\phi}_{,1}$

To calculate $\boldsymbol{\phi}_{,1}$ presents the familiar closure problem of calculating third-order velocity correlations $\langle \boldsymbol{u}(\boldsymbol{k})^* N(\boldsymbol{k})\rangle$ in terms of velocity covariances (stresses). A relatively simple, and fairly accurate, approximation for this closure was given for the case of neutral stratification. Here, we need only extend that calculation to include stratification. This approximation is obtained by expressing $\boldsymbol{u}$ in terms of a second-order velocity correlation so that $\langle \boldsymbol{u}(\boldsymbol{k})^* N(\boldsymbol{k})\rangle$ can be expressed as a two-time, fourth-order velocity correlation. A 'mild' kind of closure approximation can be applied directly to that fourth-order correlation.

To begin this closure, the desired expression for $\boldsymbol{u}$ is obtained by a formal integration of (2), the Navier–Stokes equation:

$$\boldsymbol{u}(t) = \boldsymbol{u}(0) - \int_0^t \mathrm{d}t_1 \left[(\boldsymbol{u}\cdot\nabla \boldsymbol{u})' + \frac{\nabla p}{\rho_0} + \boldsymbol{u}\cdot\nabla \boldsymbol{U} + \boldsymbol{U}\cdot\nabla \boldsymbol{u} + \frac{\boldsymbol{g}\theta}{\theta_0}\right], \tag{8}$$

where it is understood that $\boldsymbol{u}, p, \theta$ and $\boldsymbol{U}$ in the integrand are at time t_1 [e.g. $\boldsymbol{u} \equiv \boldsymbol{u}(t_1)$ in the integrand], and the molecular viscosity term $\nu\nabla^2\boldsymbol{u}$ has been neglected. This neglect can be justified at high Reynolds number since, then, the scales influenced by the viscous terms are too small to contribute significantly to the pressure–strain term. The Fourier transform of (8) is taken, and, in addition, (5) is substituted for $p(\boldsymbol{k})$. The result is

$$\boldsymbol{u}(\boldsymbol{k}, t) = \boldsymbol{u}(\boldsymbol{k}, 0) - \int_0^t \mathrm{d}t_1 \left[(\boldsymbol{u}\cdot\nabla \boldsymbol{u})_{\boldsymbol{k}} + \mathrm{i}\boldsymbol{k}N(\boldsymbol{k}) + \left(\boldsymbol{g} - \frac{\boldsymbol{g}\cdot\boldsymbol{k}^2}{k^2}\right)\frac{\theta}{\Theta_0} + \boldsymbol{h}\right], \tag{9}$$

where $(\boldsymbol{u}\cdot\nabla\boldsymbol{u})_k$ denotes the Fourier transform of $(\boldsymbol{u}\cdot\nabla\boldsymbol{u})'$

$$(\boldsymbol{u}\cdot\nabla\boldsymbol{u})_k \equiv \int \mathrm{d}\boldsymbol{x}(\boldsymbol{u}\cdot\nabla\boldsymbol{u})' \exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}),$$

and $\boldsymbol{h}$ denotes all the terms containing U. We do not trouble to evaluate $\boldsymbol{h}$ here because we estimate its contribution to $\boldsymbol{\Phi}_{,1}$ is quite small in a uniformly moving frame of reference (cf. Weinstock 1981 *a*) and will be neglected in the next paragraph. The term $(\boldsymbol{g}\cdot\boldsymbol{k}^2/k^2)\,\theta/\Theta_0$ comes from the $\mathrm{i}\boldsymbol{g}\cdot\boldsymbol{k}\theta/\Theta_0$ term of $p(\boldsymbol{k})$ given in (5).

A simplified expression for $\boldsymbol{\Phi}_{,1}$ can now be derived by substitution of (9) into the $\langle \boldsymbol{u}^* N\rangle$ term on the right-hand side of (7):

$$\boldsymbol{\Phi}_{,1} \equiv -\frac{\mathrm{i}}{V}\int \frac{\mathrm{d}\boldsymbol{k}\boldsymbol{k}}{(2\pi)^3}\langle \boldsymbol{u}(\boldsymbol{k},t)^* N(\boldsymbol{k},t)\rangle, \tag{10a}$$

$$\langle \boldsymbol{u}(\boldsymbol{k})^* N(\boldsymbol{k})\rangle = \int_0^t \mathrm{d}t_1 \Big\{\langle[(\boldsymbol{u}\cdot\nabla\boldsymbol{u})_k + \mathrm{i}\boldsymbol{k}N(\boldsymbol{k},t_1)]^* N(\boldsymbol{k},t)\rangle$$
$$+\Theta_0^{-1}\Big(\boldsymbol{g}-\frac{\boldsymbol{g}\cdot\boldsymbol{k}^2}{k^2}\Big)\langle\theta(\boldsymbol{k},t_1)^* N(\boldsymbol{k},t)\rangle\Big\}, \quad (t \gg k^{-1}v_0^{-1}) \tag{10b}$$

where $\boldsymbol{h}$, the term containing U, has been neglected as small, v_0 is the r.m.s. turbulence velocity and, in addition, the initial-value term $\langle \boldsymbol{u}(\boldsymbol{k},0)^* N(\boldsymbol{k},t)\rangle$ has been neglected since when divided by $\langle \boldsymbol{u}(\boldsymbol{k},t)^* N(\boldsymbol{k},t)\rangle$ it decays towards zero as t increases beyond the Eulerian decay time $(kv_0)^{-1}$, (Weinstock 1981 *a*).

The first average quantity in the integral on the right-hand side of (10*b*) is a fourth-order correlation, and, when substituted in (10*a*), yields the 'tendency-towards-isotropy' term, the term sometimes approximated by the model suggested by Rotta (1951). The second average quantity in (10*b*) is a correction which arises from buoyancy. This buoyancy term is new. It is the term we wish to calculate. The first term (the 'return-towards-isotropy' term) is not new. In the absence of buoyancy, this term was calculated in much detail (Weinstock 1981 *a*, 1982; Weinstock & Burk 1985), and its matrix components ij were found to be given approximately by

$$-\frac{\mathrm{i}}{V}\int\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\int \mathrm{d}t_1\, k_i\langle[(\boldsymbol{u}\cdot\nabla u_j)_k + \mathrm{i}k_j N]^* N(\boldsymbol{k},t)\rangle = -\tfrac{1}{2}C_{ij}^{(1)}\frac{\epsilon}{e_0}b_{ij}, \tag{11}$$

where e_0 is the kinetic energy density of random fluctuations ($e_0 = \frac{3}{2}v_0^2$), ϵ is the rate of dissipation of kinetic energy by molecular viscosity, and $b_{ij} \equiv \langle u_i u_j\rangle - \frac{2}{3}e_0\delta_{ij}$ is the stress anisotropy. [This expression is discussed in the cited references, and we shall not discuss it here except to point out that it differs from the Rotta model in that the numerical coefficients $C_{ij}^{(1)}$ are not the same for all i and j (i.e. $C_{11}^{(1)} \neq C_{22}^{(1)} \neq C_{33}^{(1)}$), and, in addition, vary with $\langle u_i u_j\rangle$ in a manner determined by the theory.] We obtain the same expression for the unstratified case under the assumption that the shape of the scalar velocity spectrum is not affected by buoyancy when the Reynolds number is very large. In both cases (stratified and unstratified) we assume an inertial subrange with a $k^{-\frac{5}{3}}$ power law and, at larger scales, a $k^m(m > -1)$ power law. What is allowed to vary with buoyancy is the anisotropy of kinetic energy density. Substitution of (11) into the ij-component of (10*a*), we have $\phi_{ij,1}$, the ij component of $\boldsymbol{\Phi}_{,1}$ given by

$$\phi_{ij,1} = -\tfrac{1}{2}C_{ij}^{(1)}\frac{\epsilon}{e_0}b_{ij} + \frac{\mathrm{i}}{V}\int\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\int_0^t \mathrm{d}t_1\, B_{ij}\langle\theta(\boldsymbol{k},t_1)^* N(\boldsymbol{k},t)\rangle \tag{12}$$

$$B_{ij} \equiv \Theta_0^{-1}(k_i g_j - k_i k_j \boldsymbol{g}\cdot\boldsymbol{k}/k^2)$$

where B_{ij} is used for convenience of notation.

There now remains to calculate the 'fluctuation field' buoyancy term on the right-hand side of (12). This is done by use of the thermodynamic equation (the potential temperature equation) to express $\langle \theta^* N \rangle$ as a fourth-order cumulant of the form $\boldsymbol{g} \cdot \langle \theta^* \boldsymbol{u}^* \boldsymbol{uu} \rangle \cdot \boldsymbol{k}^2$, followed by use of our (mild) cumulant discard approximation. This calculation of the $\langle \theta^* N \rangle$ term in (12) is very similar to that previously given for the case of no-buoyancy (Weinstock 1981*a*), with an additional approximation for the influence of buoyancy on Eulerian timescales. For the interested reader, the derivation is given in Appendix A to show how buoyancy enters in detail. We emphasize, as strongly pointed out by a referee, that the derivation is limited to slowly varying mean fields. The derived result is

$$(\phi_{ij} + \phi_{ji})_1 = -\frac{C_{ij}^{(1)} \epsilon b_{ij}/e_0}{(1+\frac{1}{2}H/F)^{\frac{1}{2}}} - (C_{1\theta} + C_{1\theta}^0)\,(P_{ij}^\theta - \tfrac{2}{3} P^\theta \delta_{ij}), \tag{13}$$

$$P_{ij}^\theta \equiv -\Theta_0^{-1}(g_i \langle u_j \theta \rangle + g_j \langle u_i \theta \rangle),$$

$$P^\theta \equiv \tfrac{1}{2} P_{ii}^\theta, \quad \text{(summed on } i\text{)}$$

$$F \equiv 14 \epsilon^2 e_0^{-2} \omega_B^{-2},$$

$$C_{1\theta} \cong 0.4(1+\tfrac{1}{2}H/F)^{-1},$$

$$C_{1\theta}^0 \equiv \phi_{22,1}(FP^\theta)^{-1}(1+\tfrac{1}{2}H/F)^{-1},$$

$$H \equiv \begin{cases} 1, & \omega_B^2 > 0, \\ 0, & \omega_B^2 \leqslant 0, \end{cases}$$

where P_{ij}^θ is the buoyancy production (or loss) of energy, $\omega_B \equiv (-\theta_0^{-1} \boldsymbol{g} \cdot \nabla \theta_0)^{\frac{1}{2}}$ is the Brunt–Väisälä frequency, F is a dimensionless buoyancy parameter referred to as the Froude number, $C_{1\theta}$ is the theoretical coefficient calculated in an approximate way from first principles, with the value 0.4 for very large Reynolds number, $\boldsymbol{g}$ is assumed to be directed along the x_2-coordinate, and $H \equiv H(\omega_B)$, the Heaviside step function, occurs because the $\frac{1}{2}F^{-1}$ correction vanishes when the stratification is unstable ($\omega_B^2 < 0$). Equation (13) gives $(\phi_{ij} + \phi_{ji})_1$, the fluctuation part of the pressure–strain term, in the presence of buoyancy. It is seen that this part of the pressure–strain term tends to isotropize the buoyancy production P_{ij}^θ, similarly to the mean field part (Launder 1975; Zeman & Lumley 1976) shown in the next section. The quantity $C_{1\theta}^0$ is a dimensionless parameter whose magnitude can be shown to be less than $C_{1\theta}$ (about one third $C_{1\theta}$) for a nearly uniform mean shear flow (Webster 1964; Tavoularis & Corrsin 1981). One may ignore $C_{1\theta}^0$ for practical purposes.

A limitation of (13) pointed out by a referee is its implication that a sudden change in the near-field quantity Θ_0 is reflected immediately in the fluctuation field, whereas, in reality, it takes a little time for the fluctuation field to equilibrate with the mean field. This defect in (13) is a consequence of our basic restriction to mean fields that vary slowly in time compared to the Lagrangian timescale i.e. (13) applies only when the mean fields are slowly varying. This restriction is imposed in going from (12) to (13). The integral in (12) contains the time history (memory) of the mean field and fluctuations, and this time history is ignored in the derivation of (13). Further discussion of this point is given in §6.

4. $\Phi_{,3}$ and completion of the pressure–strain term

4.1. *The mean-field buoyancy term*

The mean-field buoyancy term $\Phi_{,3}$ was previously calculated by Launder (1975) and Zeman & Lumley (1976). Both found

$$(\phi_{ij}+\phi_{ji})_3 = -C_{3\theta}(P^{\theta}_{ij}-\tfrac{2}{3}P^{\theta}\,\delta_{ij}), \tag{14}$$

where $C_{3\theta}$ is a numerical constant. Zeman & Lumley obtained $C_{3\theta} = \frac{3}{10}$ (hereinafter referred to as the theoretical value of $C_{3\theta}$) by an approximate solution of the integral for $\Phi_{,3}$ in (7), and Launder (1975) obtained $C_{3\theta} \cong 0.6$ from an empirical consideration (these two values are reconciled in §4.3). Upon comparison, it is seen that (14) is the same form as the buoyancy correction to $(\phi_{ij}+\phi_{ji})_1$ given by (13), and, further, the magnitudes are nearly the same. Since the two buoyancy corrections have nearly the same magnitude both must be included, and, since they have the same form, only one numerical coefficient is needed. This is discussed next for the total pressure–strain term. An applicability difference between the two buoyancy corrections is that (14) is valid for rapidly as well as slowly varying mean fields whereas (13) is *a priori* limited to slowly varying fields since a 'memory' term was neglected in its derivation.

4.2. *Completion of the pressure–strain term*

The total pressure–strain term is given by substitution of (13) and (14) in (7):

$$\left\langle \frac{p}{\rho_0}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right\rangle = -\frac{C^{(1)}_{ij}\,\epsilon b_{ij}/e_0}{(1+\frac{1}{2}H/F)^{\frac{1}{2}}}+(\phi_{ij}+\phi_{ji})_2-C^{*}_{\theta}(P^{\theta}_{ij}-\tfrac{2}{3}P^{\theta}\,\delta_{ij}), \tag{15}$$

$$C^{*}_{\theta} \equiv C_{1\theta}+C^{0}_{1\theta}+C_{3\theta}.$$

This equation is our principal substantive result. The principal influence of buoyancy is given by the third term on the right-hand side of (15). In addition, there is also a minor buoyancy correction to the 'resistance-to-large anisotropy' term, the first term on the right-hand side, which is significant when $F \approx 1$. Roughly speaking, F is proportional to the ratio of turbulence kinetic energy to the kinetic energy of gravity wave fluctuations. Consequently, F is significant in (15), when the energy in waves is comparable to the energy in turbulence. The kinetic energy $e_0 \equiv \frac{1}{2}\langle \boldsymbol{u}'\cdot\boldsymbol{u}'\rangle$ includes gravity waves as well as random turbulence since $\boldsymbol{u}'$ is the total velocity of random fluctuations.

With regard to the mean shear term $(\phi_{ij}+\phi_{ji})_2$ in (15), an approproximate expression was derived by Launder *et al.* (1975) and Naot, Shavit & Wolfshstein (1973) as

$$(\phi_{ij}+\phi_{ji})_2 = -(c_2+8)\{P_{ij}-\tfrac{2}{3}P\delta_{ij}\}/11-(30c_2-2)\,e_0\{\partial U_i/\partial x_j+\partial U_j/\partial x_i\}/55$$
$$-(8c_2-2)\{D_{ij}-\tfrac{2}{3}P\delta_{ij}\}/11,$$

where $P_{ij} \equiv -\{\langle u_i u_k\rangle\,\partial U_j/\partial x_k+\langle u_j u_k\rangle\,\partial U_i/\partial x_k\}$ is the mechanical production term which occurs in the Reynolds stress equation, $P \equiv \frac{1}{2}P_{ii}$, and $D_{ij} \equiv -\{\langle u_j u_k\rangle\,\partial U_k/\partial x_i+\langle u_i u_k\rangle\,\partial U_k/\partial x_j\}$. We have found this expression satisfactory for both strongly and weakly sheared nearly homogeneous unstratified flows when c_2 is selected to be 0.42 (see Weinstock & Burk 1985).

4.3. *Reconciliation of two previous theories of* $\Phi_{,3}$

The buoyancy term, the third term on the right-hand side of (15), is seen to be fairly simple, an isotropizing buoyancy production term multiplied by a single numerical coefficient C_θ^*. As we previously showed, only this coefficient was needed since the buoyancy contribution to $\phi_{,1}$ has the same form as $\phi_{,3}$, provided that the mean field does not vary too rapidly, and, therefore, the sum of $\phi_{,1}$ and $\phi_{,3}$ could be expressed in terms of one coefficient. Hence, there is no error of practical consequence when $\phi_{,3}$ is retained while the buoyancy correction to $\phi_{,1}$ is neglected, provided that the numerical coefficient $C_{3\theta}$ is chosen to be sufficiently large to compensate for the neglect of $C_{1\theta}$. This procedure can be ascribed to Launder (1975), who, in effect, chose $C_{1\theta} = 0$ and $C_{3\theta} = 0.6$, with the resultant C_θ^* equal to 0.6, a choice fortuitously quite close to the theoretical value of $C_{1\theta}+C_{3\theta}$. We thus find a reconciliation between the numerical coefficient of Zeman & Lumley (1976) and that of Launder (1975), i.e. the theoretical coefficient $C_{3\theta}$ is approximately $\frac{3}{10}$ as calculated by Zeman & Lumley, and, at the same time, the larger coefficient suggested by Launder should be used for turbulence modelling since $C_{1\theta}$ occurs in addition to $C_{3\theta}$.

5. Experimental evaluation of C_θ^*

A remaining consideration is to determine if the theoretical value $C_\theta^* \approx 0.7$ is accurate at large F, And, if not, to obtain a more accurate value of C_θ^*. This must be done by comparison with experiment. The experiment of Webster (1964) appears most suitable to determine C_θ^*, although there is doubt (by Webster, himself) that his data attained equilibrium.

Since experiments observe the relative stress anisotropy m_{ij}, we must, in order to compare theory with experiment, determine how the theoretical C_θ^* influences m_{ij}. An approximate theoretical expression for m_{ij} is readily obtained from the Reynolds stress equation (Rodi 1976 – algebraic modelling). That equation is given, for a steady state, by

$$\frac{\mathrm{D}\langle u_i u_j\rangle}{\mathrm{D}t} = P_{ij}+P_{ij}^\theta+\left\langle\frac{p}{\rho_0}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial u_i}\right)\right\rangle-\epsilon_{ij}. \tag{16}$$

The pressure–strain term is given by (15). For a quasi-uniform shear flow with U along the x_1-direction and the gradient along x_2, the Launder *et al.* (1975) model can be expressed as

$$(\phi_{ij}+\phi_{ji})_2 = -C_{ij}^{(2)}(P_{ij}-\tfrac{2}{3}P\delta_{ij}), \tag{17}$$

where $C_{11}^{(2)} = 0.73$, $C_{22}^{(2)} = 0.53$, $C_{33}^{(2)} = 0.92$, $C_{12}^{(2)} = 0.6$ for the choice $c_2 = 0.42$ (this value of c_2 is suggested by Weinstock & Burk 1985). Substitution of (15) and (17) into (16), and setting $\mathrm{D}\langle u_i u_j\rangle/\mathrm{D}t$ equal to zero for the Webster experiment, we solve for the relative anisotropy $m_{ij} \equiv e_0^{-1}(\langle u_i u_j\rangle-\tfrac{2}{3}e_0\delta_{ij})$ to obtain

$$m_{ij} = \frac{(1-C_{ij}^{(2)})(P_{ij}-\tfrac{2}{3}P\delta_{ij})+(1-C_\theta^*)(P_{ij}^\theta-\tfrac{2}{3}P\delta_{ij})-(\epsilon_{ij}-\tfrac{2}{3}\epsilon\delta_{ij})}{C_{ij}^{(1)}\epsilon}, \tag{18}$$

where the term $\frac{1}{2}H/F$ is ignored for Websters' experiment since, there, $F \gg 1$. By use of $P_{11} = 2P$ and $P = \epsilon-P^\theta$, the diagonal components of m_{ij} can be written as follows:

$$\left.\begin{aligned}
m_{11} &= \left[\tfrac{4}{3}(1-C_{11}^{(2)})+(\tfrac{4}{3}C_{11}^{(2)}+\tfrac{2}{3}C_\theta^*-2)\frac{P^\theta}{\epsilon}-\left(\frac{\epsilon_{11}}{\epsilon}-\frac{2}{3}\right)\right](C_{11}^{(1)})^{-1},\\
m_{22} &= \left[-\tfrac{2}{3}(1-C_{22}^{(2)})+(2-\tfrac{2}{3}C_{22}^{(2)}-\tfrac{4}{3}C_\theta^*)\frac{P^\theta}{\epsilon}-\left(\frac{\epsilon_{22}}{\epsilon}-\frac{2}{3}\right)\right](C_{22}^{(1)})^{-1},\\
m_{33} &= \left[-\tfrac{2}{3}(1-C_{33}^{(2)})+\tfrac{2}{3}(C_\theta^*-C_{33}^{(2)})\frac{P^\theta}{\epsilon}-\left(\frac{\epsilon_{33}}{\epsilon}-\frac{2}{3}\right)\right](C_{33}^{(1)})^{-1}.
\end{aligned}\right\} \tag{19}$$

It is seen that the theoretical m_{33} varies with P^{θ}/ϵ as $(C_{33}^{(1)})^{-1}\frac{2}{3}(C_{\theta}^{*}-C_{33}^{(2)})\,P^{\theta}/\epsilon$. This quantity must be very close to zero in order to agree with the experimental data which show that m_{33} varies very little with $P^{\theta}/\epsilon = R_{\mathrm{f}}$ (Webster 1964; see figure 1 of Launder 1975). Hence,

$$C_{\theta}^{*} \approx C_{33}^{(2)}. \tag{20}$$

This equation yields $C_{\theta}^{*} \approx C_{33}^{(2)} \approx 0.9$ when c_2 is chosen to be 0.42 in the Launder *et al.* (1975) model of $C_{33}^{(2)}$. The reason why Launder (1975) obtained the coefficient 0.6 from the same data is that he employed the abridged model of $(\phi_{ij}+\phi_{ij})_2$, and that model has the coefficients $C_{ij}^{(2)} = c_2 = 0.6$ (all i and j). Consequently, when substituted in (20) the abridged model yields $C_{\theta}^{*} = 0.6$.

The experimental value $C_{\theta}^{*} \approx 0.9$ is not conclusive because the observed stresses did not reach equilibrium. It does differ from the theoretical value of about 0.6 to 0.7, but the difference is within approximation errors of the theory.

6. Summary and discussion

A theoretical calculation was made of the buoyancy contribution to $(\phi_{ij}+\phi_{ji})_1$, the slow-term part of the pressure–strain term. This contribution was found to have the same form as the mean-field buoyancy term $(\phi_{ij}+\phi_{ji})_3$ previously calculated by Launder (1975) and Zeman & Lumley (1976), provided that the mean buoyancy field varies slowly in time and space. Because of their similarity, both buoyancy contributions are accounted for by use of a single numerical coefficient C_{θ}^{*}, the sum of the coefficients of the two terms. The combined buoyancy contribution to the pressure–strain term is $C_{\theta}^{*}(P_{ij}^{\theta}-\frac{2}{3}P\delta_{ij})$. The theoretical value of C_{θ}^{*} is about 0.7, and the value estimated from the experiment of Webster (1964) is 0.9. The experimental value is a little uncertain since the measured stresses may not have reached equilibrium. Such uncertainty notwithstanding, the 25 % difference between experimental and theoretical C_{θ}^{*} lends support to the theoretical approximations made in our Appendix. The value $C_{\theta}^{*} = 0.9$ obtained from Webster (1964) could be verified by further experiment in nearly homogeneous turbulent shear flow operated at values of R_{f} in excess of about 0.2.

For $R_{\mathrm{f}} > 0$, the present theory implicitly includes fluctuations of gravity waves in addition to turbulence, i.e. e_0 is the total energy in fluctuations, turbulence plus waves. Roughly speaking, energy scales smaller than the buoyancy length $L_{\mathrm{B}} \equiv 2\pi\epsilon^{\frac{1}{2}}N^{-\frac{3}{2}}$ are viewed as turbulence whereas energy scales exceeding L_{B} are viewed as gravity waves. Not specified or discussed is the distinction between random and coherent gravity waves. One suspects that random waves will influence the pressure–strain term differently from coherent waves, but a theory to account for this difference has not been worked out.

A question touching on the broader aspects of turbulence modelling is implied by the memory term in the integrand of (12). This term implies that the structure of turbulence depends on the auto-correlation time, the Eulerian timescale, of the fluctuation field. This memory is not usually included in turbulence models. Hence, one may wonder if such turbulence models, based on slow variations and weak inhomogeneities, are meaningful when the variations are rapid. Lumley (1978) has discussed this question in a general way. The reader is referred to that discussion. All we have to add is that the influence of the memory term may be significantly weakened when the turbulence equations are integrated forward in time, since, then, the auto-correlation and mean field in (12) occur in the integrand of a double time integral. This double integral tends to 'smooth out' rapid changes. The extent of smoothing can be estimated by utilizing (12), instead of (13), in test turbulence modelling computations, with substitution of various possible timescale variations for $\Theta_0(t_1)$ and $\langle\Theta(\boldsymbol{k},t_1)^* N(\boldsymbol{k},t)\rangle$.

Appendix A

To derive (13) from (12), the calculation of $\langle\theta^* N\rangle$, we begin with the thermodynamic equation

$$\frac{\partial\theta}{\partial t}+(\boldsymbol{u}+\boldsymbol{U})\cdot\nabla\theta-\langle\boldsymbol{u}\cdot\nabla\theta\rangle+\boldsymbol{u}\cdot\nabla\Theta_0=0, \quad \text{(A 1)}$$

where molecular conductivity has been neglected for the scales to be considered. Straight-forward integration yields

$$\theta(t_1)=\theta(0)-\int_0^{t_1}\mathrm{d}t_2[(\boldsymbol{u}\cdot\nabla\theta)'+\boldsymbol{u}\cdot\nabla\Theta_0+\boldsymbol{U}\cdot\nabla\theta], \quad \text{(A 2)}$$

where $(\boldsymbol{u}\cdot\nabla\theta)'\equiv(\boldsymbol{u}\cdot\nabla\theta)-\langle\boldsymbol{u}\cdot\nabla\theta\rangle$, and it is understood that $\boldsymbol{u}=\boldsymbol{u}(t_2)$, $\theta=\theta(t_2)$, etc. in the integrand. Substituting the Fourier transform of (A 2) into $\langle\theta^* N\rangle$, and neglecting $\boldsymbol{U}\cdot\nabla\theta$, gives us

$$\langle\theta(\boldsymbol{k},t_1)^* N(\boldsymbol{k},t)\rangle\equiv\langle\theta(\boldsymbol{k},0)^* N(\boldsymbol{k},t)\rangle--\int_0^{t_1}\mathrm{d}t_2[\langle(\boldsymbol{u}\cdot\nabla\theta)_{\boldsymbol{k}}^* N(\boldsymbol{k},t)\rangle$$
$$+\langle(\boldsymbol{u}(\boldsymbol{k},t)^* N(\boldsymbol{k},t)\rangle\cdot\nabla\Theta_0], \quad \text{(A 3)}$$

and $(\boldsymbol{u}\cdot\nabla\theta)_{\boldsymbol{k}}$ denotes the Fourier transform of $(\boldsymbol{u}\cdot\nabla\theta)'$; it is explicitly given by (A 6). The neglected $\boldsymbol{U}\cdot\nabla\theta$ term in (A 3) is estimated to be small, similarly to the neglect of the $\boldsymbol{U}$ terms in (10*b*). Also similar to (10*b*), the initial-value term in (A 3) decays towards zero when t exceeds $(kv_0)^{-1}$, and be neglected for large enough t. Substitution of (A 3) in (12) gives

$$\phi_{ij,1}=-\tfrac{1}{2}C_{ij}^{(1)}(\epsilon/e_0)\,b_{ij}$$
$$-\frac{\mathrm{i}}{V}\int\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\int_0^t\mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\,B_{ij}[\underbrace{\langle(\boldsymbol{u}\cdot\nabla\theta)_{\boldsymbol{k}}^* N(t)\rangle}_{\beta_{ij}}+\underbrace{\langle\boldsymbol{u}(\boldsymbol{k},t_2)^* N(t)\rangle\cdot\nabla\Theta_0}_{\beta_{ij}^0}]. \quad \text{(A 4)}$$

This equation shows two buoyancy corrections to $\phi_{ij,1}$, a fourth-order correlation β_{ij} and a third-order correlation β_{ij}^0. However, we find that the fourth-order correlation is the larger of the two for the experimental conditions of Webster (1964), by a factor of about three. We calculate β_{ij} fist. We simplify the notation by letting F_θ denote the fourth-order correlation in the integrand of β_{ij}:

$$F_\theta\equiv\langle(\boldsymbol{u}\cdot\nabla\theta)_{\boldsymbol{k}}^* N(t)\rangle, \quad \text{(A 5)}$$

and then substitute (6) for $N(\boldsymbol{k},t)$, and, also substitute

$$(\boldsymbol{u}\cdot\nabla\theta)_{\boldsymbol{k}}^*=(2\pi)^{-3}\int\mathrm{d}\boldsymbol{k}_\mathrm{b}\,\boldsymbol{u}(\boldsymbol{k}_\mathrm{b},t_2)^*\cdot(-\mathrm{i})\,(\boldsymbol{k}-\boldsymbol{k}_\mathrm{b})\,\theta(\boldsymbol{k}-\boldsymbol{k}_\mathrm{b},t_2)^*, \quad \text{(A 6)}$$

it being understood that $(\boldsymbol{u}\cdot\nabla\theta)$ in (A 4) is a function of t_2. We thus have F_θ expressed as

$$F_\theta=\mathrm{i}\int\frac{\mathrm{d}\boldsymbol{k}_\mathrm{a}}{(2\pi)^3}\int\frac{\mathrm{d}\boldsymbol{k}_\mathrm{b}}{(2\pi)^3}\,\boldsymbol{k}\cdot\langle\boldsymbol{u}(\boldsymbol{k}_\mathrm{b},t_2)^*\,\theta(\boldsymbol{k}'',t_2)^*\,[\boldsymbol{u}(\boldsymbol{k}_\mathrm{a},t)\,\boldsymbol{u}(\boldsymbol{k}',t)]'\rangle:\frac{\boldsymbol{k}^2}{k^2}, \quad \text{(A 7)}$$

$$\boldsymbol{k}'\equiv\boldsymbol{k}-\boldsymbol{k}_\mathrm{a},\quad \boldsymbol{k}''\equiv\boldsymbol{k}-\boldsymbol{k}_\mathrm{b}, \quad \text{(A 8)}$$

where we have used incompressibility

$$\boldsymbol{k}_{\mathrm{b}}\cdot\boldsymbol{u}(\boldsymbol{k}_{\mathrm{b}},t_2) = 0, \tag{A 9}$$

and the definition $[\boldsymbol{uu}]' \equiv \boldsymbol{uu}-\langle\boldsymbol{uu}\rangle$. To close the fourth-order correlation in (A 7) we expand it in cumulants as

$$\langle \boldsymbol{u}(\boldsymbol{k}_{\mathrm{b}},t_2)^*\,\theta(\boldsymbol{k}'',t_2)^*\,[\boldsymbol{u}(\boldsymbol{k}_{\mathrm{a}},t)\,\boldsymbol{u}(\boldsymbol{k}',t)]'\rangle = \langle \boldsymbol{u}(\boldsymbol{k}_{\mathrm{b}},t_2)^*\,\boldsymbol{u}(\boldsymbol{k}_{\mathrm{a}},t)\rangle\,\langle\theta(\boldsymbol{k}'',t_2)^*\,\boldsymbol{u}(\boldsymbol{k}',t)\rangle$$
$$+\langle \boldsymbol{u}(\boldsymbol{k}_{\mathrm{b}},t_2)^*\,\boldsymbol{u}(\boldsymbol{k}',t)\rangle\,\langle\theta(\boldsymbol{k}'',t_2)^*\,\boldsymbol{u}(\boldsymbol{k}_{\mathrm{a}},t)\rangle + Q_\theta^{(4)}(t;t_2), \tag{A 10}$$

where $Q_\theta^{(4)}(t;t_2)$ is the fourth-order cumulant of the correlation on the left-hand side of (A 10). Of special importance here, we note that $Q_\theta^{(4)}(t;t_2)$ is a 'two-time' cumulant and is very small for large $(t-t_1)$ (i.e. for $t-t_1 > (kv_0)^{-1}$). Our basic approximation is to neglect $Q_\theta^{(4)}$. Neglect of this two-time correlation is not as serious as the neglect of 'single-time' fourth-order cumulants in quasi-normal theory (e.g. Proudman & Reid 1954). A similar neglect of two-time cumulants is basic to the direct interaction approximation (Kraichman 1959).

We express the second-order correlations of (A 10) in terms of the velocity spectrum $\boldsymbol{S}$ defined by $\boldsymbol{S}(\boldsymbol{k};t,t_2) \equiv \langle \boldsymbol{u}(\boldsymbol{k},t_2)^*\,\boldsymbol{u}(\boldsymbol{k},t)\rangle\,V^{-1}$ and the mixed spectrum (temperature flux) $\boldsymbol{R}(\boldsymbol{k};t,t_2) \equiv \langle\theta(\boldsymbol{k},t_2)^*\,\boldsymbol{u}(\boldsymbol{k},t)\rangle\,V^{-1}$ as follows,

$$\left.\begin{aligned}\langle \boldsymbol{u}(\boldsymbol{k}_{\mathrm{b}},t_2)^*\,\boldsymbol{u}(\boldsymbol{k}_{\mathrm{a}},t)\rangle &= \boldsymbol{S}(\boldsymbol{k}_{\mathrm{a}};t,t_2)\,(2\pi)^3\,\delta(\boldsymbol{k}_{\mathrm{b}}-\boldsymbol{k}_{\mathrm{a}})\\ \langle\theta(\boldsymbol{k}'',t)^*\,\boldsymbol{u}(\boldsymbol{k}',t)\rangle &= \boldsymbol{R}(\boldsymbol{k}';t,t_2)\,(2\pi)^3\,\delta(\boldsymbol{k}''-\boldsymbol{k}'),\end{aligned}\right\} \tag{A 11}$$

where δ is the Dirac delta function, and $\boldsymbol{S}$ and $\boldsymbol{R}$ have been normalized with volume V so as to satisfy the normalization condition

$$\left.\begin{aligned}\int\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\,\boldsymbol{S}(\boldsymbol{k};t,t_2) &= \langle \boldsymbol{u}(\boldsymbol{r},t)^*\,\boldsymbol{u}(\boldsymbol{r},t_2)\rangle,\\ \int\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\,\boldsymbol{R}(\boldsymbol{k};t,t_2) &= \langle \boldsymbol{u}(\boldsymbol{r},t)^*\,\theta(\boldsymbol{r},t_2)\rangle.\end{aligned}\right\} \tag{A 12}$$

Equation (A 11) is valid for homogeneous turbulence, and approximately so for our quasi-homogeneous case of slow variations of average quantities on scales $2\pi k^{-1} \leqslant L_0$. Substituting (A 11) in (A 7), and using (A 8), we obtain

$$F_\theta(\boldsymbol{k};t,t_2) = 2\mathrm{i}\,V\int\frac{\mathrm{d}\boldsymbol{k}_{\mathrm{a}}}{(2\pi)^3}\,\boldsymbol{k}\cdot\boldsymbol{R}(\boldsymbol{k}';t,t_2)\,\boldsymbol{S}(\boldsymbol{k}_{\mathrm{a}};t,t_2):\frac{\boldsymbol{k}^2}{k^2}, \tag{A 13}$$

which expresses F_θ in terms of covariances.

The desired buoyancy term β_{ij}, the principal contribution of buoyancy to $\phi_{ij,1}$, is now given by substitution of (A 13) and (A 5) into β_{ij} defined by (A 4)

$$\beta_{ij} = 2\int\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\int\frac{\mathrm{d}\boldsymbol{k}_{\mathrm{a}}}{(2\pi)^3}\int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\,B_{ij}\,\boldsymbol{k}\cdot\boldsymbol{R}(\boldsymbol{k}';t,t_2)\,\boldsymbol{S}(\boldsymbol{k}_{\mathrm{a}};t,t_2):\frac{\boldsymbol{k}^2}{k^2}. \tag{A 14}$$

Next, we express the right-hand side in terms of single-point covariances (Reynolds stresses), and single-point temperature flux $\langle\theta(\boldsymbol{r},t)\,\boldsymbol{u}(\boldsymbol{r},t)\rangle$. This is readily accomplished since there is sufficient knowledge available about the behaviour of $\boldsymbol{S}(\boldsymbol{k};t,t_2)$ when $t \neq t_2$, and a closely related calculation has already been made (Weinstock 1981*a*) for the unstratified case. We need only account for the influence of buoyancy. To begin with we use

$$\int_0^t \mathrm{d}t_1\int_0^t \mathrm{d}t_2\,\boldsymbol{R}(\boldsymbol{k}';t,t_2)\,\boldsymbol{S}(\boldsymbol{k}_{\mathrm{a}};t,t_2) = [(\tau_{k'})^{-1}+(\tau_{k_{\mathrm{a}}})^{-1}]^{-2}\,\boldsymbol{R}(\boldsymbol{k}';t,t)\,\boldsymbol{S}(\boldsymbol{k}_{\mathrm{a}};t,t) \tag{A 15}$$

where τ_k is an Eulerian integral time, and was discussed by Lumley *et al.* (1978) and Weinstock (1978, 1981*b*). It is approximately given by

$$\tau_k = [k^2 v_k^2 + \tfrac{1}{2}\omega_B^2 H(\omega_B)]^{-\frac{1}{2}}, \qquad \text{(A 16)}$$

$$\omega_B^2 \equiv -(\boldsymbol{g}/\Theta_0)\cdot\nabla\Theta_0,$$

$$H(\omega_B) \equiv \begin{cases} 1, & \omega_B^2 > 0 \\ 0, & \omega_B^2 \leqslant 0. \end{cases}$$

Here, ω_B is the Brunt–Väisälä frequency, H is the Heaviside step function (this function arises because unstable stratification does not influence the correlation decay time in a direct or obvious fashion), and v_k^2 is the kinetic energy density residing in random fluctuations whose wavenumber is less than k; i.e. $v_k^2 = \int_k^\infty \mathrm{d}k'\, E(k')$ where $E'(k')$ is the scalar kinetic energy density of random fluctuations. Equation (A 16) is an assumed model to account for the damping of velocity fluctuations by stable stratification, the decrease of integral timescales with increase of ω_B. Its virtue is that it is approximately correct in the limits of large kv_k/ω_B and small kv_k/ω_B (for positive ω_B) (see Weinstock 1981*b*).

Substituting (A 15) into (A 14) we have

$$\beta_{ij} = 2\int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\int \frac{\mathrm{d}\boldsymbol{k}_a}{(2\pi)^3} B_{ij}\tau_c^2\, \boldsymbol{k}\cdot\boldsymbol{R}(\boldsymbol{k}')\,\boldsymbol{S}(\boldsymbol{k}_a):\frac{\boldsymbol{k}^2}{k^2}. \qquad \text{(A 17)}$$

$$\tau_c^2 \equiv [(\tau_{k'})^{-1} + (\tau_{k_a})^{-1}]^{-2}.$$

The $\boldsymbol{k}$ and $\boldsymbol{k}_a$ integrations can be performed by making fairly weak assumptions about the behaviour of $\boldsymbol{R}(\boldsymbol{k}')$ and $\boldsymbol{S}(\boldsymbol{k}_a)$ (Weinstock 1981*a*). However, much simplification is achieved by approximating $\boldsymbol{S}$ with its isotropic part: $\boldsymbol{S}^{\mathrm{I}}(\boldsymbol{k}_a) = 2\pi^2(\boldsymbol{I} - \boldsymbol{k}_a^2/k_a^2)\,E(k_a)/k_a^2)$ where $\boldsymbol{I}$ is the identity matrix. Substituting $\boldsymbol{S}^{\mathrm{I}}$ into (A 17) and use of the definition $\boldsymbol{k}\cdot\boldsymbol{R} = \boldsymbol{k}\cdot\langle \boldsymbol{u}(\boldsymbol{k}')^*\,\theta(\boldsymbol{k}')\rangle\, V^{-1}$ we have

$$\beta_{ij} = \frac{2}{V}\int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\int \frac{\mathrm{d}k_a}{4\pi} B_{ij}\tau_c^2\,\frac{E(k_a)}{k_a^2}\left[1 - \frac{(\boldsymbol{k}\cdot\boldsymbol{k}_a)^2}{k^2k_a^2}\right]\boldsymbol{k}\cdot\langle \boldsymbol{u}(\boldsymbol{k}')^*\,\theta(\boldsymbol{k}')\rangle. \qquad \text{(A 18)}$$

To simplify the integrations in (A 18), we specify $\boldsymbol{g}$ to be directed along the x_2-axis. This introduces no loss of generality in (A 18), since the integrations are over all $\boldsymbol{k}$ and $\boldsymbol{k}_a$ and we can choose the **2** direction arbitrarily. Further simplification is possible because (A 18) implies

$$\beta_{11} = \beta_{33} = -\tfrac{1}{2}\beta_{22}, \qquad \text{(A 19)}$$

since

$$B_{11} = -k_2(k_1^2/k^2)\,g_2/\Theta_0, \quad B_{33} = -k_2(k_3^2/k^2)\,g_2/\Theta_0,$$

$$B_{22} = k_2(1 - k_2^2/k^2)\,g_2/\Theta_0 = \kappa_2(\kappa_1^2/\kappa^2 + \kappa_3^2/\kappa^2)\,g_2/\Theta_0,$$

and in the integration, k_1 can be approximately interchanged with k_3. Hence, we need only calculate one diagonal element to obtain the others. We choose to calculate β_{22}. Anticipating the fact that in (A 18) the main contribution comes from the $k_2\langle u_2(\boldsymbol{k}')^*\,\theta(\boldsymbol{k}')\rangle$ component of $\boldsymbol{k}\cdot\langle \boldsymbol{u}(\boldsymbol{k}')^*\,\theta(\boldsymbol{k}')\rangle$, the other two components having been found to be much smaller, we multiply numerator and denominator of (A 18) by

$$\left(\frac{g_2}{\Theta_0}\right)\langle u_2\,\theta\rangle \equiv (2\pi)^{-3}V^{-1}\left(\frac{g_2}{\theta_0}\right)\int \mathrm{d}\boldsymbol{k}\langle u_2(\boldsymbol{k})^*\,\theta(\boldsymbol{k})\rangle,$$

and rewrite (A 18) for $i = j = 2$ as

$$\beta_{22} = \tfrac{2}{3} C_{1\theta} \frac{g_2}{\Theta_0} \langle u_2 \theta \rangle, \tag{A 20}$$

$$C_{1\theta} \equiv 3 \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3} \int \frac{\mathrm{d}\boldsymbol{k}_a}{(2\pi)^3} B_{22} \tau_c^2 \frac{E(k_a)}{k_a^2} \left[1 - \frac{(\boldsymbol{k}\cdot\boldsymbol{k}_a)^2}{k^2 k_a^2} \right] \boldsymbol{k}\cdot\langle \boldsymbol{u}(\boldsymbol{k}')^* \theta(\boldsymbol{k}')\rangle \times \left[\frac{g_2}{\Theta_0} \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3} \langle u_2(\boldsymbol{k})^* \theta(\boldsymbol{k})\rangle \right]^{-1}. \tag{A 21}$$

Here, $C_{1\theta}$ can be seen to be a non-dimensional coefficient and, in face, is a weighted mean value averaged over the spectrum $\langle \boldsymbol{u}(\boldsymbol{k})^* \theta(\boldsymbol{k})\rangle$. Because it is such an average, $C_{1\theta}$ is not particularly sensitive to the spectral shapes of $\langle u_2(\boldsymbol{k}')^* \theta(\boldsymbol{k}')\rangle$ and $E(k_a)$, and, consequently, approximate spectral shapes can be used in (A 21). A similar use of approximate spectra to calculate the pressure–strain coefficients $C_{ij}^{(1)}$ were shown to be accurate to within 20 % (Weinstock 1981*a*). The spectrum $E(k)$ is approximated as done previously (Weinstock 1981*a*) by $E(k) = \alpha\epsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$ for $k_0 \leqslant k \leqslant k_\nu$, and $E(k) = \alpha\epsilon^{\frac{2}{3}}(k_0^{-m-\frac{5}{3}})k^m$ for $k < k_0$, where k_ν is the viscous 'cut-off' wavenumber, $m > -1$ is an adjustable parameter, and k_0 is the wavenumber where the spectrum is maximum. The approximation for $\langle u_2(\boldsymbol{k})\theta(\boldsymbol{k})\rangle$ is to neglect its variations with the direction of $\boldsymbol{k}$ (its angular variations) and to represent its scalar variations by $\langle u_2(\boldsymbol{k})^* \theta(\boldsymbol{k})\rangle = ak^{-\frac{5}{3}}$ for $k_0 \leqslant k \leqslant k_\nu$, and $\langle u_2(\boldsymbol{k})^* \theta(\boldsymbol{k})\rangle = a(k_0^{-m-\frac{5}{3}})k^m$ for $k < k_0$, where a is constant coefficient which need not be specified. In actuality, experimental data of Kaimal *et al.* (1972) suggest that the spectrum $\langle u_2(\boldsymbol{k})\theta(\boldsymbol{k})\rangle \propto k^{-\frac{7}{3}}$ at very large k, but that there is an intermediate range where the spectrum is very broad and could be approximated by $k^{-\frac{5}{3}}$. Of course, the data are not necessarily pertinent because they were for a surface layer where wall effects are important, whereas wall effects are ignored in our calculations.

With this approximation for $\langle u_2(\boldsymbol{k})^* \theta(\boldsymbol{k})\rangle$, and use of our approximation that $\boldsymbol{k}\cdot\langle \boldsymbol{u}(\boldsymbol{k}')^* \theta(\boldsymbol{k}')\rangle \cong k_2 \langle u_2(\boldsymbol{k}')^* \theta(\boldsymbol{k}')\rangle$ for the integration in (A 21), all the quantities in (A 21) are specified and the integration can be performed straightforwardly for various values of m, k_0, and k_ν. It is then found that (A 21) is very insensitive to values of m, suggesting a 'universal' behaviour for $C_{1\theta}$. The calculated value of $C_{1\theta}$ is

$$C_{1\theta} + 0.4(1 + \tfrac{1}{2} k_0^{-2} v_0^{-2} \omega_B^2 H)^{-1}(1 - R_\nu^{-\frac{1}{4}}), \tag{A 22}$$

where the 'Reynolds number' $R_\nu \cong v_0/\nu k_0$ comes from the viscous 'cut-off' at wavenumber k_ν, and the $(k_0^2 v_0^2)^{-1} \omega_B^2$ term comes from the buoyancy-dependent decay time τ_c defined in (A 17) and (A 16).

The other diagonal elements β_{11} and β_{33} are determined by (A 19) and (A 20) to be $-\frac{1}{3} C_{1\theta} g_2 \langle u_2 \theta \rangle$. Hence, the diagonal components of β_{ij} can generally be expressed as

$$\beta_{ij} + \beta_{ji} = -C_{1\theta}(P_{ij}^\theta - \tfrac{2}{3} P^\theta \delta_{ij}) \quad (i = j) \tag{A 23}$$

$$P_{ij}^\theta \equiv -\Theta_0^{-1}[g_i \langle u_j \theta \rangle + g_j \langle u_i \theta \rangle],$$

$$P^\theta \equiv \tfrac{1}{2}\,\mathrm{Trace}\, P_{ij}^\theta,$$

where P_{ij}^θ is the buoyancy production (or loss) term of the Reynolds stress equation. Although (A 23) was derived for $\boldsymbol{g} = -g_2 \hat{\boldsymbol{x}}_2$, it can be shown to be valid for an arbitrary direction of $\boldsymbol{g}$.

The off-diagonal element $\beta_{ij} + \beta_{ji}$ is calculated from (A 18) by anticipating the fact that, for this element, the component $k_1 \langle u_1(\boldsymbol{k}')^* \theta(\boldsymbol{k}')\rangle$ is more important in (A 18) than the other components of $\boldsymbol{k}\cdot\langle \boldsymbol{u}(\boldsymbol{k}')^*\,\theta(\boldsymbol{k}')\rangle$. We then multiply the numerator and denominator of (A 18) by $(g_2/\Theta_0)\langle u_1\,\theta\rangle \equiv (2\pi)^{-3}V^{-1}\int \mathrm{d}\boldsymbol{k}\langle u_1(\boldsymbol{k})^*\,\theta(\boldsymbol{k})\rangle$ and rewrite the result as

$$\beta_{12}+\beta_{21} = C'_{1\theta}\left(\frac{g_2}{\Theta_0}\right)\langle u_1\,\theta\rangle \equiv -C'_{1\theta}P^{\theta}_{12}, \tag{A 24}$$

$$C'_{1\theta} \equiv 2\int\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\int\frac{\mathrm{d}\boldsymbol{k}_{\mathrm{a}}}{(2\pi)^3}(B_{12}+B_{21})\,\tau_{\mathrm{c}}^2\,\frac{E(k_{\mathrm{a}})}{k_{\mathrm{a}}^2}\left[1-\frac{(\boldsymbol{k}\cdot\boldsymbol{k}_{\mathrm{a}})^2}{k^2k_{\mathrm{a}}^2}\right]\boldsymbol{k}\cdot\langle \boldsymbol{u}(\boldsymbol{k}')^*\,\theta(\boldsymbol{k}')\rangle$$
$$\times\left[\frac{g_2}{\Theta_0}\int\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\langle u_1(\boldsymbol{k})^*\,\theta(\boldsymbol{k})\rangle\right]^{-1}, \tag{A 25}$$

where $C'_{1\theta}$ is a numerical constant. The integration of (A 25) for $C'_{1\theta}$ is performed in much the same way as was done for (A 21), except that here, we use the experimental fact that $\langle u_1(\boldsymbol{k})^*\,\theta(\boldsymbol{k})\rangle$ varies as $k^{-\frac{5}{2}}$, instead of $k^{-\frac{5}{3}}$ at short scales, the inertial subrange scales (Kaimal *et al.* 1972). We thus obtain

$$C'_{1\theta} \approx 0.6(1+k_0^{-2}v_0^{-2}\,\omega_{\mathrm{B}}^2\,H)^{-1}(1-R_\nu^{-\frac{1}{4}}). \tag{A 26}$$

It is seen that $C'_{1\theta}$ is $\frac{3}{2}$ as large as $C_{1\theta}$, the coefficient of the diagonal elements. If $C_{1\theta}$ and $C'_{1\theta}$ were approximately equal, the combination of (A 23) and (A 24) would give us

$$\beta_{ij}+\beta_{ji} = -C_{1\theta}(P^{\theta}_{ij}-\tfrac{2}{3}P^{\theta}\,\delta_{ij}) \quad (\text{all } i,j), \tag{A 27}$$

for all values of i and j. Approximation (A 27) is appropriate for us to make despite the difference in the values of $C_{1\theta}$ and $C'_{1\theta}$ because these values are only approximations to begin with. Equation (A 27) gives us the fourth-order correlation term that appears in (A 4), that is, the symmetric part of that term.

Next we briefly calculate β^0_{ij}, the third-order correlation in (A 4). Our goal is to obtain an approximate expression for it in terms of $\phi_{ij,\,1}$. We first use

$$\int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\langle \boldsymbol{u}(\boldsymbol{k},t_2)\,N(\boldsymbol{k},t)\rangle \approx \tau_k^2\langle \boldsymbol{u}(\boldsymbol{k},t)\,N(\boldsymbol{k},t)\rangle, \tag{A 28}$$

as was done similarly in (A 15),to write β^0_{ij} in the approximate form

$$\beta^0_{ij} = -\frac{\mathrm{i}}{V}\int\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}B_{ij}\,\tau_k^2\langle \boldsymbol{u}(\boldsymbol{k})^*\,N(\boldsymbol{k})\rangle\cdot\boldsymbol{\nabla}\Theta_0. \tag{A 29}$$

Next, we substitute (10*a*), the definition of $\phi_{ij,\,1}$, with $B_{ij} = k_i\,g_j-k_i\,k_j\,\boldsymbol{k}\cdot\boldsymbol{g}/k^2$ in (A 29), and integrate to obtain the approximation

$$\beta^0_{ij} \approx -\tau_{k_0}^2[\omega_{\mathrm{B}j}^2\,\phi_{ij,\,1}-\tfrac{1}{3}\omega_{\mathrm{B}n}^2\,\phi_{nn,\,1}\,\delta_{ij}] \quad (\text{summed on } n), \tag{A 30}$$

$$\omega_{\mathrm{B}j}^2 \equiv -\frac{g_j}{\Theta_0}\frac{\partial\Theta_0}{\partial x_j},$$

$$\tau_{k_0}^2 \equiv (k_0^2v_0^2+\tfrac{1}{2}\omega_{\mathrm{B}}^2\,H)^{-1} = \left(\frac{14\epsilon^2}{e_0^2}+\tfrac{1}{2}\omega_{\mathrm{B}}^2\,H\right)^{-1},$$

where we have used the following approximations: (*a*) the main contribution to (A 29) from the $k_i\,k_j\,k^{-2}\boldsymbol{g}\cdot\boldsymbol{k}/\Theta_0$ part of B^0_{ij} is for $i = j$ and is approximately $(\frac{1}{3})\,\boldsymbol{g}\cdot\boldsymbol{k}/\Theta_0$ in (A 29); and (*b*) the main contribution from $g_i\,\boldsymbol{\nabla}\Theta_0$ is for the $g_i\,\partial\Theta_0/\partial x_i$ component. Equation (A 30) gives the third-order correlation.

This expression for β^0_{ij} can be readily related to β_{ij} in (A 27) if we specify the direction of $\boldsymbol{g}$. Thus, we take $\boldsymbol{g}$ to be directed along x_2. We then find that (the symmetric part of) (A 30) can be expressed as

$$\beta^0_{ij}+\beta^0_{ji} = -\frac{\phi_{22,\,1}}{(F+\frac{1}{2}H)\,P^\theta}\,(P^\theta_{ij}-\tfrac{2}{3}P^\theta\,\delta_{ij}), \tag{A 31}$$

where F is a dimensionless parameter, a Froude number, given by

$$F^{-1} \equiv k_0^{-2}\,v_0^{-2}\,\omega_{\mathrm{B}}^2 \equiv 14\epsilon^2 e_0^{-2}\,\delta_{\mathrm{B}}^2.$$

Incidentally, we note that for a nearly uniform shear flow (e.g. Webster 1964; Tavoularis & Corrsin 1981), F^{-1} is approximately equal to the 'gradient' Richardson number $Ri \equiv \omega_{\mathrm{B}}^2(\partial U_0/\partial x_2)^{-2}$ since, for a such an experiment $\tau^2_{k_0} \approx (\partial U_0/\partial x_2)^{-2}$. In that case $\epsilon/[(F+\frac{1}{2}H)\,P^\theta]$ is proportional to the Prandtl number since $P^\theta = -\epsilon R_{\mathrm{f}}(1-R_{\mathrm{f}})^{-1}$, where R_{f} is the 'flux' Richardson number (the ratio of buoyancy production to shear production). That is, for a nearly uniform mean shear, we could express $(F+\frac{1}{2}H)\,P^\theta$ as

$$\frac{\epsilon}{(F+\frac{1}{2}H)\,P^\theta} \approx \frac{Ri(1-R_{\mathrm{f}})}{R_{\mathrm{f}}} = \sigma_{\mathrm{T}}(1-R_{\mathrm{f}}) \quad (F \gg 1), \tag{A 32}$$

where σ_{T} is the ratio Ri/R_{f}.

Finally, the buoyancy dependent part of $(\phi_{ij}+\phi_{ji})_1$ is given by combining (A 27) with (A 31),

$$(\beta_{ij}+\beta_{ji})+(\beta^0_{ij}+\beta^0_{ji}) = -\left(C_{1\theta}+\frac{\phi_{22,\,1}}{(F+\frac{1}{2}H)\,P^\theta}\right)(P^\theta_{ij}-\tfrac{2}{3}P^\theta\,\delta_{ij}), \tag{A 33}$$

so that

$$(\phi_{ij,\,1}+\phi_{ji,\,1}) = -C^{(1)}_{ij}\,\frac{\epsilon}{e_0}\,b_{ij}-\left(C_{1\theta}+\frac{\phi_{22,\,1}}{(F+\frac{1}{2}H)\,P^\theta}\right)(P^\theta_{ij}-\tfrac{2}{3}P^\theta\,\delta_{ij}). \tag{A 34}$$

Another buoyancy correction occurs in the resistance-to-large-anisotropy term, the first term on the right-hand side of (A 34), since $C^{(1)}_{ij}$ is theoretically proportional to $\tau_{k_0} = (k_0\,v_{k_0})^{-1}$ in the unstratified case ($\omega_{\mathrm{B}} = 0$), (see I) and, can be shown to be proportional to $\tau_{k_0} = (k_0\,v_{k_0})^{-1}\,(1+\frac{1}{2}\omega_{\mathrm{B}}^2\,k_0^{-2}\,v_k^{-2}\,H)^{-\frac{1}{2}} = (k_0\,v_{k_0})^{-1}\,(1+\frac{1}{2}H/F)^{-\frac{1}{2}}$ in the presence of stratification. Thus, we should make the replacement

$$C^{(1)}_{ij} \rightarrow C^{(1)}_{ij}(1+\tfrac{1}{2}H/F)^{-\frac{1}{2}} \tag{A 35}$$

in (A 34). Equation (A 34), with (A 35), is the fluctuation part of the pressure–strain term in the presence of buoyancy. It completes our goal of deriving (13). A simplification of (A 34) is sometimes possible since $\phi_{22,\,1}/[(F+\frac{1}{2}H)\,P^\theta]$ is smaller than $C_{1\theta}$ by about $\frac{1}{3}$ for the experimental nearly uniform shear flows of Webster (1964) and Tavoularis & Corrsin (1981) and could be ignored.

A final note about the derivation of (A 34) is that e_0 includes random fluctuations of gravity waves energy as well as turbulence energy. Not included in e_0 is the kinetic energy of coherent gravity waves. A unified treatment that includes coherent as well as random gravity waves requires more work.

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